

Convex Shape Preservation of Cubic L^1 Spline Fits

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Abstract Cubic L^1 spline fits have been developed for geometric data approximation and shown excellent performances in shape preservation. To quantify the convex-shape-preserving capability of spline fits, we consider a basic shape of convex corner with two line segments in a given window. Given the horizontal length difference and slope change of convex corner, we conduct an analytical approach and a numerical procedure to calculate the second-derivative-based spline fit in 3-node and 5-node windows, respectively. Results in both cases show that the convex shape can be well preserved when the horizontal length difference is within the middle third of window's length. In addition, numerical results in the 5-node window indicate that the second-derivative based and first-derivative based spline fits outperform function-value based spline fits for preserving this convex shape. Our study extends current quantitative research on shape preservation of cubic L^1 spline fits and provides more insights on improving advance spline node positions for shape preserving purpose.

1 Introduction

For the interpolation and approximation of geometric data with sudden changes, L^1 -norm-based functions, in various forms, have been shown to provide superior

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shape preservation, such as L^1 interpolating splines [1,3,10,12,16,21], L^1 spline fits [11,18] and L_1 approximation from Chebyshev and weak-Chebyshev spaces [7]. The minimization principles of cubic L^1 interpolating splines, also embedded in constructing L^1 spline fits, can be based on the second derivatives [8], first derivatives [9,12] and function values [13], respectively. In literatures, global algorithms and local moving-window methods have been designed for producing spline fits that preserve shapes well [8,9,18,20]. We have seen applications of cubic L^1 spline fits in urban terrain reconstruction [2,15] and other engineering areas such as robust control of robots [14,17], sensor-collected data analytics [5], large data sets approximation [6] and surface reconstruction [4].

The first step towards theoretical analysis on the shape-preserving capability of cubic L^1 spline fits defined a shape-preserving metric for a basic shape of two parallel line segments [19]. The length difference between the line segments in a given window was regarded as the extra information that leads the spline fits to preserve the linear shape. Both the analytic approach in a 3-node window and a numerical procedure in a 5-node window showed that the spline fits have convincingly strong capability in preserving linear shape.

Moving one step further, we propose to theoretically analyze the shape-preserving capability of cubic L^1 spline fits for a convex shape. The principle of preserving convex shape is that, if some portion of data is aligned along a convex curve, the spline fit should preserve the convexity of that portion. We consider a convex corner, which is a simple and commonly seen pattern in irregular geometric data, for instance, the shape formed by a flat ground and its ground ascent. Note that the spline nodes are usually beforehand fixed in practical applications. Hence, the relative positions of spline nodes may have significant impacts on the shape-preserving capability of a spline fit. In addition, the slope of two line segments, which characterizes the convex shape at corner, may also be an important factor to the shape-preserving capability of the spline fit.

In this study, we consider a convex corner consisting of a horizontal line on the left and a slope line on the right. The second-derivative based spline fit is applied to approximate this basic shape in a window of odd-number equally spaced spline nodes. Note that the horizontal length difference and slope change of two line segments fully characterize the convex corner. We explore the conditions on these two parameters under which the spline fits remain to be convex in the given window.

To calculate the spline fit of convex corner with given horizontal length difference and slope change, we conduct an analytic approach in a 3-node window and a numerical procedure in a 5-node window. In the 3-node window, boundary conditions are imposed such that the information of convex corner can be contained at the boundary. We solve the spline interpolating problem and then spline fit problem analytically. We can calculate the function value and first derivative at the middle spline node, based on which the convexity of spline fit is further checked. Our analytical results show that the convexity of spline fit can be well preserved when the horizontal length difference is within the middle third interval of window's length, but is independent with the slope change of two line segments. In the 5-node window, we solve the spline fit problem numerically. We set the slope change and keep adjusting the horizontal length difference by a given precision. No boundary conditions are fixed in this case.

Numerical results show that the spline fit is convex in the 5-node window if and only if the horizontal length difference is with the middle third of interval's length. Results in 3-node and 5-node windows are consistent in characterizing the performance of second-derivative-based cubic L^1 spline fits for preserving the convex shape. This study is the first to quantitatively analyze the convex-shape-preserving performance of second-derivative-based spline fit. It can be used as a reference to improve spline node positions of spline fit for convex-shape preservation purpose and to predict the approximating results of spline fit in various circumstances.

Meanwhile, we adopt numerical procedures to compare the shape-preserving performance of three types of spline fits in the 5-node window. Our numerical results show that the second-derivative based and first-derivative based spline fits are better in preserving the shape of convex corner. For shape preservation purpose, the function-value based spline fit is not recommended for the approximation of convex corner. To fully evaluate the convex-shape-preserving capability of three types of spline fits, it is worth finding analytic solutions to the first-derivative based and function-value based spline fits for the convex corner in the 3-node window, and then define a convex-shape-preserving metric accordingly.

The remaining of this paper is organized as follows. In Sect. 2, we introduce a convex corner and cubic L^1 spline fits for a function. In Sect. 3, we calculate the second-derivative based spline fit for a convex corner in a 3-node window with fixed boundary conditions. In Sect. 4, we numerically compute the second-derivative based spline fit for a convex corner in a 5-node window. Section 5 compares the shape-preserving performance of three types of spline fits in the 5-node window. In Sect. 6, we summarize the paper with some remarks.

2 Spline Fits for a Convex Corner

We investigate the shape-preserving capability of cubic L^1 spline fits for the convex shape shown in Fig. 1. Without loss of generality, we may assume that the line segment on the left is horizontally longer than that on the right. $\epsilon > 0$ denotes the extra information of horizontal line and α is the slope angle with $0^\circ < \alpha < 90^\circ$. This convex corner has the following mathematical formula:

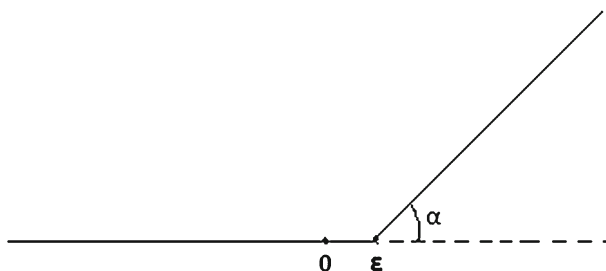


Fig. 1 A convex corner

$$f(x) = \begin{cases} 0, & \text{if } x \leq \epsilon, \\ k(x - \epsilon), & \text{if } x \geq \epsilon, \end{cases}$$

where $k = \tan \alpha > 0$ is the slope change. Denote the monotonically increasing spline nodes as $x_i, i = 0, 1, \dots, I$. Let the function values and first derivatives of spline fit at these nodes be z_i and b_i , respectively. Define $\mathbf{z} = (z_0, z_1, \dots, z_I)$, $\mathbf{b} = (b_0, b_1, \dots, b_I)$, $h_i = x_{i+1} - x_i$ and $\Delta z_i = \frac{z_{i+1} - z_i}{h_i}$. A C^1 -smooth piecewise cubic polynomials $z(x)$ with nodes $x_0, x_1, \dots, x_I$ takes the form of

$$z(x) = z_i + b_i(x - x_i) + \frac{1}{h_i}[-(2b_i + b_{i+1}) + 3\Delta z_i](x - x_i)^2 + \frac{1}{h_i^2}[b_i + b_{i+1} - 2\Delta z_i](x - x_i)^3$$

in each interval of $[x_i, x_{i+1}]$, $i = 0, 1, \dots, I$. A cubic L^1 spline fit of function $f(x)$ is a piecewise cubic function $z(x)$ with nodes $x_0, x_1, \dots, x_I$ that minimizes

$$F(\mathbf{z}, \mathbf{b}) = \int_{x_0}^{x_I} |z(x) - f(x)| dx$$

over the manifold of cubic L^1 interpolating splines, that is, C^1 -smooth piecewise cubic polynomials $z(x)$ that minimize an L^1 interpolating functional as described below:

- The second-derivative-based interpolating functional is

$$G(\mathbf{b}) = \int_{x_0}^{x_I} |z''(x)| dx$$

with given nodal values $z_0, z_1, \dots, z_I$.

- The first-derivative-based interpolating functional is

$$G(\mathbf{b}) = \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{1}{h_i} |z'(x) - \zeta'(x)| dx,$$

where $\zeta(x)$ is the piecewise linear function that connects $(x_0, z_0), \dots, (x_I, z_I)$, defined as

$$\zeta(x) = z_i + \frac{\Delta z_i}{h_i}(x - x_i). \quad (1)$$

- The function-value-based interpolating functional is

$$G(\mathbf{b}) = \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{1}{h_i^2} |z(x) - \zeta(x)| dx$$

with $\zeta(x)$ the same as (1).

3 Second-Derivative-Based Spline Fit in a 3-Node Window

We consider the second-derivative-based spline fit in a 3-node window where $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Note that the 3-node window is relatively small, hence, we impose boundary conditions so that the information of original shape can be contained at the boundary. Specifically, we set

$$z_0 = 0, \quad b_0 = 0, \quad z_2 = (1 - \epsilon)k \text{ and } b_2 = k, \quad (2)$$

which means that the boundary nodes coincide with the original shape with respect to function values and first derivatives.

Then, the cubic spline function can be expressed based on (z_1, b_1) as follows:

$$z(x) = \begin{cases} (-b_1 + 3z_1)(x+1)^2 + (b_1 - 2z_1)(x+1)^3, & \text{if } x \in [-1, 0], \\ z_1 + b_1x + [-2b_1 + (2 - 3\epsilon)k - 3z_1]x^2 + [b_1 - (1 - 2\epsilon)k + 2z_1]x^3, & \text{if } x \in [0, 1]. \end{cases} \quad (3)$$

Accordingly, the first-order derivative and second-order derivative functions are given below.

$$z'(x) = \begin{cases} 2(-b_1 + 3z_1)(x+1) + 3(b_1 - 2z_1)(x+1)^2, & \text{if } x \in [-1, 0], \\ b_1 + 2[-2b_1 + (2 - 3\epsilon)k - 3z_1]x + 3[b_1 - (1 - 2\epsilon)k + 2z_1]x^2, & \text{if } x \in [0, 1], \end{cases} \quad (4)$$

and

$$z''(x) = \begin{cases} 2(-b_1 + 3z_1) + 6(b_1 - 2z_1)(x+1), & \text{if } x \in [-1, 0], \\ 2[-2b_1 + (2 - 3\epsilon)k - 3z_1] + 6[b_1 - (1 - 2\epsilon)k + 2z_1]x, & \text{if } x \in [0, 1]. \end{cases} \quad (5)$$

Calculating the second-derivative-based L^1 spline fit for $f(x)$ requires solving the following bi-level optimization problem:

$$\min_{z_1} \int_{-1}^1 |z(x) - f(x)| dx,$$

where $z(x)$ is the second-derivative-based interpolating spline in the form of (3) defined by z_1 and $b_1^*(z_1)$:

$$b_1^*(z_1) = \arg \min_{b_1} \int_{-1}^1 |z''(x)| dx.$$

3.1 Interpolating Spline Problem

We first solve the interpolating spline problem $\min_{b_1} G(b_1)$, where

$$G(b_1) = \int_{-1}^1 |z''(x)| dx. \quad (6)$$

Plugging $z''(x)$ in the form of (5) into (6), we have $G(b_1) = G_1(b_1) + G_2(b_1)$ with

$$\begin{aligned} G_1(b_1) &= \int_{-1}^0 |z''(x)| dx = \theta(-z_1, b_1 - z_1), \\ G_2(b_1) &= \int_0^1 |z''(x)| dx = \theta(b_1 + z_1 - (1 - \epsilon)k, z_1 + \epsilon k), \end{aligned}$$

where the bivariate function of $\theta(p, q)$ is defined by

$$\theta(p, q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |(q - p) + 6t(p + q)| dt.$$

Following Lemma 3 in [1] and Lemma 2 in [8], the explicit expression of $G_1(b_1)$ is given by

$$G_1(b_1) = \begin{cases} |b_1|, & \text{if } |b_1| \geq 3|b_1 - 2z_1|, \\ \frac{3}{2}|b_1 - 2z_1| + \frac{b_1^2}{6|b_1 - 2z_1|}, & \text{else.} \end{cases}$$

Let $\delta_1^{(1)}$ and $\delta_2^{(1)}$ be solutions to the equation of $|b_1| = 3|b_1 - 2z_1|$ in terms of b_1 . Without loss of generality, we may assume that $\delta_1^{(1)} \leq \delta_2^{(1)}$. Then,

$$G'_1(b_1) = \begin{cases} -\frac{5}{3} + \frac{1}{6} \left(\frac{2z_1}{b_1 - 2z_1} \right)^2, & \text{if } b_1 \in (-\infty, \delta_1^{(1)}), \\ \text{sgn}(z_1), & \text{if } b_1 \in [\delta_1^{(1)}, \delta_2^{(1)}], \\ \frac{5}{3} - \frac{1}{6} \left(\frac{2z_1}{b_1 - 2z_1} \right)^2, & \text{if } b_1 \in (\delta_2^{(1)}, \infty), \end{cases} \quad (7)$$

where $\text{sgn}(\cdot)$ is the sign function.

Remark 1 When $z_1 = 0$, we have $\delta_1^{(1)} = \delta_2^{(1)} = 0$. Then $G(b_1)$ becomes non-differentiable at $b_1 \in [\delta_1^{(1)}, \delta_2^{(1)}]$. In this case, $\text{sgn}(z_1) = 0$ is a subgradient of $G(b_1)$ at $b_1 \in [\delta_1^{(1)}, \delta_2^{(1)}]$.

Similarly, we have

$$G_2(b_1) = \begin{cases} |b_1 - k|, & \text{if } |b_1 - k| \geq 3|(b_1 - k) + 2(z_1 + \epsilon k)|, \\ \frac{3}{2} |(b_1 - k) + 2(z_1 + \epsilon k)| + \frac{(b_1 - k)^2}{6|(b_1 - k) + 2(z_1 + \epsilon k)|}, & \text{else,} \end{cases}$$

and

$$G'_2(b_1) = \begin{cases} -\frac{5}{3} + \frac{1}{6} \left[\frac{2(z_1 + \epsilon k)}{(b_1 - k) + 2(z_1 + \epsilon k)} \right]^2, & \text{if } b_1 \in \left(-\infty, \delta_1^{(2)} \right), \\ -\text{sgn}(z_1 + \epsilon k), & \text{if } b_1 \in \left[\delta_1^{(2)}, \delta_2^{(2)} \right], \\ \frac{5}{3} - \frac{1}{6} \left[\frac{2(z_1 + \epsilon k)}{(b_1 - k) + 2(z_1 + \epsilon k)} \right]^2, & \text{if } b_1 \in \left(\delta_2^{(2)}, \infty \right), \end{cases} \quad (8)$$

where $\delta_1^{(2)}$ and $\delta_2^{(2)}$ are solutions to the equation of $|b_1 - k| = 3|(b_1 - k) + 2(z_1 + \epsilon k)|$ in terms of b_1 with $\delta_1^{(2)} \leq \delta_2^{(2)}$.

Based on the forms of $G'_1(b_1)$ and $G'_2(b_1)$, the next theorem provides the solution that minimizes the interpolating functional $G(b_1) = G_1(b_1) + G_2(b_1)$.

Theorem 1 Given $z_0 = 0, b_0 = 0, z_2 = (1 - \epsilon)k$ and $b_2 = k$ with $\epsilon \in (0, 1)$ and $k > 0$, the first derivative b_1^* that minimizes the second-derivative-based L^1 interpolating functional $G(b_1)$ of form (6) is a function of z_1 and has the expression as below.

- When $\epsilon \in (0, \frac{1}{3})$:

$$b_1^*(z_1) = \begin{cases} \frac{kz_1}{2z_1 + \epsilon k}, & \text{if } z_1 \in (-\infty, -\epsilon k] \cup [0, \frac{1}{6}(1 - 3\epsilon)k) \cup [\frac{1}{6}(2 - 3\epsilon)k, \infty), \\ \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}, & \text{if } z_1 \in (-\epsilon k, 0), \\ b_1 \in [(1 - 3\epsilon)k - 3z_1, 3z_1], & \text{if } z_1 \in [\frac{1}{6}(1 - 3\epsilon)k, \frac{2}{9}(1 - 3\epsilon)k), \\ b_1 \in [\frac{3}{2}z_1, (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1], & \text{if } z_1 \in [\frac{2}{9}(1 - 3\epsilon)k, \frac{1}{6}(2 - 3\epsilon)k). \end{cases} \quad (9)$$

- When $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$:

$$b_1^*(z_1) = \begin{cases} \frac{kz_1}{2z_1 + \epsilon k}, & \text{if } z_1 \in (-\infty, -\epsilon k] \cup [\frac{1}{6}(2 - 3\epsilon)k, \infty), \\ \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}, & \text{if } z_1 \in (-\epsilon k, \frac{1}{4}(1 - 3\epsilon)k], \\ z_1, & \text{if } z_1 \in (\frac{1}{4}(1 - 3\epsilon)k, 0), \\ b_1 \in [\frac{3}{2}z_1, 3z_1], & \text{if } z_1 \in [0, \frac{2}{9}(1 - \frac{3}{2}\epsilon)k), \\ b_1 \in [\frac{3}{2}z_1, (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1], & \text{if } z_1 \in [\frac{2}{9}(1 - \frac{3}{2}\epsilon)k, \frac{1}{6}(2 - 3\epsilon)k). \end{cases} \quad (10)$$

- When $\epsilon \in (\frac{2}{3}, 1)$:

$$b_1^*(z_1) = \begin{cases} \frac{kz_1}{2z_1 + \epsilon k}, & \text{if } z_1 \in (-\infty, -\epsilon k] \cup [0, \infty), \\ \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}, & \text{if } z_1 \in (-\epsilon k, \frac{1}{4}(1 - 3\epsilon)k], \\ z_1, & \text{if } z_1 \in (\frac{1}{4}(1 - 3\epsilon)k, \frac{1}{5}(2 - 3\epsilon)k], \\ g(z_1), & \text{if } z_1 \in (\frac{1}{5}(2 - 3\epsilon)k, 0), \end{cases} \quad (11)$$

where $g(\cdot)$ is a function such that $b_1^* = g(z_1)$ solves $\left(\frac{2z_1}{b_1^* - 2z_1}\right)^2 + \left(\frac{2(z_1 + \epsilon k)}{(b_1^* - k) + 2(z_1 + \epsilon k)}\right)^2 = 10$ for $z_1 \in (\frac{1}{5}(2 - 3\epsilon)k, 0)$.

Proof From (7) and (8), we see that both G'_1 and G'_2 are increasing functions in terms of $b_1 \in \mathbb{R}$. Then $G(b_1) = G_1(b_1) + G_2(b_1)$ is convex with respect to b_1 . Any solution of $b_1^*(z_1)$ satisfying $G'_1(b_1^*) + G'_2(b_1^*) = 0$ will solve the interpolating spline problem.

The expression of $G'_1(b_1)$ depends on intervals defined by $\delta_1^{(1)}$ and $\delta_2^{(1)}$ with $\delta_1^{(1)} \leq \delta_2^{(1)}$, which are solutions to the equation of $|b_1| = 3|b_1 - 2z_1|$. Therefore, $(\delta_1^{(1)}, \delta_2^{(1)}) = (\frac{3}{2}z_1, 3z_1)$ if $z_1 \geq 0$, and $(3z_1, \frac{3}{2}z_1)$ otherwise. Similarly, $G'_2(b_1)$ depends on intervals defined by $\delta_1^{(2)}$ and $\delta_2^{(2)}$ with $(\delta_1^{(2)}, \delta_2^{(2)}) = ((1 - 3\epsilon)k - 3z_1, (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1)$ if $z_1 \geq -\epsilon k$, and $((1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1, (1 - 3\epsilon)k - 3z_1)$ otherwise. Since the realization of $G'(b_1) = G'_1(b_1) + G'_2(b_1)$ depends on the value of z_1 , we discuss the following three cases:

Case 1: $z_1 \leq -\epsilon k$. In this case, we have

$$\delta_1^{(1)} = 3z_1, \quad \delta_2^{(1)} = \frac{3}{2}z_1; \quad \delta_1^{(2)} = \left(1 - \frac{3}{2}\epsilon\right)k - \frac{3}{2}z_1, \quad \delta_2^{(2)} = (1 - 3\epsilon)k - 3z_1,$$

with $\delta_2^{(1)} \leq 0 \leq k \leq \delta_1^{(2)}$.

It can be calculated that $G'_1(\delta_1^{(2)}) \geq G'_1(\delta_2^{(1)}) = -1$ and $G'_2(\delta_2^{(1)}) \leq G'_2(\delta_1^{(2)}) = 1$. Therefore,

$$G'_1(\delta_2^{(1)}) + G'_2(\delta_1^{(1)}) \leq 0 \leq G'_1(\delta_1^{(2)}) + G'_2(\delta_1^{(2)}),$$

which further implies $b_1^* \in [\delta_2^{(1)}, \delta_1^{(2)}]$. Then, b_1^* can be obtained by solving the following equation:

$$\left\{ \frac{5}{3} - \frac{1}{6} \left(\frac{2z_1}{b_1 - 2z_1} \right)^2 \right\} + \left\{ -\frac{5}{3} + \frac{1}{6} \left[\frac{2(z_1 + \epsilon k)}{(b_1 - k) + 2(z_1 + \epsilon k)} \right]^2 \right\} = 0,$$

which is equivalent to

$$\left| \frac{2z_1}{b_1 - 2z_1} \right| = \left| \frac{2(z_1 + \epsilon k)}{(b_1 - k) + 2(z_1 + \epsilon k)} \right|. \quad (12)$$

Since $b_1^* \geq \delta_2^{(1)} \geq 2z_1$ and $b_1^* - k \leq \delta_1^{(2)} - k \leq -2(z_1 + \epsilon k)$, (12) becomes

$$-\frac{2z_1}{b_1^* - 2z_1} = \frac{2(z_1 + \epsilon k)}{(b_1^* - k) + 2(z_1 + \epsilon k)},$$

leading to

$$b_1^* = \frac{kz_1}{2z_1 + \epsilon k}.$$

Case 2: $-\epsilon k < z_1 \leq 0$. In this case, we have

$$\delta_1^{(1)} = 3z_1, \quad \delta_2^{(1)} = \frac{3}{2}z_1; \quad \delta_1^{(2)} = (1 - 3\epsilon)k - 3z_1, \quad \delta_2^{(2)} = \left(1 - \frac{3}{2}\epsilon\right)k - \frac{3}{2}z_1,$$

with $\delta_2^{(1)} \leq z_1 \leq 0$ and $\delta_2^{(2)} \leq k$.

- If $\epsilon \in (0, \frac{1}{3})$, then we further derive that $\delta_2^{(1)} \leq z_1 \leq \delta_1^{(2)} \leq \delta_2^{(2)} \leq k$. Subsequently, $G'_1(\delta_2^{(1)}) = -1$, $G'_1(\delta_1^{(2)}) \geq G'_1(z_1) = 1$ and $G'_2(\delta_2^{(1)}) \leq G'_1(\delta_1^{(2)}) = -1$, which further implies

$$G'_1(\delta_2^{(1)}) + G'_2(\delta_2^{(1)}) \leq 0 \leq G'_1(\delta_1^{(2)}) + G'_2(\delta_1^{(2)}).$$

Therefore, $b_1^* \in [\delta_2^{(1)}, \delta_1^{(2)}]$ satisfies (12). As $z_1 + \epsilon k \geq 0$, (12) becomes

$$-\frac{2z_1}{b_1^* - 2z_1} = -\frac{2(z_1 + \epsilon k)}{(b_1^* - k) + 2(z_1 + \epsilon k)},$$

and thus

$$b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}.$$

- If $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, then we discuss the following two subcases:
 - (2.1) $z_1 \in (-\epsilon k, \frac{1}{4}(1 - 3\epsilon)k]$, under which we further have $\delta_2^{(1)} \leq z_1 \leq \delta_1^{(2)} \leq \delta_2^{(2)} \leq k$. This is the same as the situation of $\epsilon \in (0, \frac{1}{3})$. Therefore, $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$.
 - (2.2) $z_1 \in (\frac{1}{4}(1 - 3\epsilon)k, 0]$, which leads to $\delta_2^{(1)} \leq \delta_1^{(2)} \leq z_1 \leq \delta_2^{(2)} \leq k$. Subsequently, $G'_2(z_1) = -1$, and thus $G'_1(z_1) + G'_2(z_1) = 0$. Since G'_1 is strictly increasing in $(\delta_2^{(1)}, \infty)$ and G'_2 is increasing in $(-\infty, \infty)$, we know that $b_1^* = z_1$ is unique.
- If $\epsilon \in (\frac{2}{3}, 1)$, then we discuss the following three subcases:
 - (2.3) $z_1 \in (-\epsilon k, \frac{1}{4}(1 - 3\epsilon)k]$, under which we have $\delta_2^{(1)} \leq z_1 \leq \delta_1^{(2)} \leq \delta_2^{(2)} \leq k$, similar as subcase (2.1). Accordingly, $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$.
 - (2.4) $z_1 \in [\frac{1}{4}(1 - 3\epsilon)k, \frac{1}{5}(2 - 3\epsilon)k]$, which leads to $\delta_1^{(2)} \leq z_1 \leq \delta_2^{(2)} \leq k$, similar as the subcase (2.2). Then, $b_1^* = z_1$.
 - (2.5) $z_1 \in [\frac{1}{5}(2 - 3\epsilon)k, 0]$, implying that $\delta_2^{(2)} \leq z_1$. Note that it turns out to be $G'_1(z_1) + G'_2(z_1) \leq 0$ and $G'_1(\delta_2^{(2)}) + G'_2(\delta_2^{(2)}) \leq 0$. Meanwhile, $G'_2(\delta_2^{(1)}) \leq G'_2(k) = 1$ follows $\delta_2^{(1)} \leq 0 \leq k$, leading to $G'_1(\delta_2^{(1)}) + G'_2(\delta_2^{(1)}) \leq 0$. Consequently, we have $b_1^* \in [\max\{\delta_2^{(1)}, \delta_2^{(2)}\}, z_1]$ and thus b_1^* is the solution to

$$\left\{ \frac{5}{3} - \frac{1}{6} \left(\frac{2z_1}{b_1^* - 2z_1} \right)^2 \right\} + \left\{ \frac{5}{3} - \frac{1}{6} \left[\frac{2(z_1 + \epsilon k)}{(b_1^* - k) + 2(z_1 + \epsilon k)} \right]^2 \right\} = 0.$$

Case: 3 $z_1 > 0$. In this case, we have

$$\delta_1^{(1)} = \frac{3}{2}z_1, \quad \delta_2^{(1)} = 3z_1; \quad \delta_1^{(2)} = (1 - 3\epsilon)k - 3z_1, \quad \delta_2^{(2)} = \left(1 - \frac{3}{2}\epsilon\right)k - \frac{3}{2}z_1,$$

- with $0 \leq \delta_1^{(1)}$.
- If $\epsilon \in (0, \frac{1}{3})$, then we discuss the following four subcases:
 - (3.1) $z_1 \in (0, \frac{1}{6}(1-3\epsilon)k]$, which implies $\delta_1^{(1)} \leq \delta_2^{(1)} \leq \delta_1^{(2)} \leq \delta_2^{(2)}$. Then, $b_1^* \in [\delta_2^{(1)}, \delta_1^{(2)}]$ satisfies (12), with $b_1^* - 2z_1 \geq 0$ and $(b_1^* - k) + 2(z_1 + \epsilon k) \leq 0$. Therefore, $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.
 - (3.2) $z_1 \in (\frac{1}{6}(1-3\epsilon)k, \frac{2}{9}(1-3\epsilon)k]$, which implies $\delta_1^{(1)} \leq \delta_1^{(2)} \leq \delta_2^{(1)} \leq \delta_2^{(2)}$. Obviously, we have $G'_1(b_1) = 1$ and $G'_2(b_1) = -1$ for $b_1 \in [\delta_1^{(2)}, \delta_2^{(1)}]$. As a result, the interval of $[\delta_1^{(2)}, \delta_2^{(1)}]$ is optimal.
 - (3.3) $z_1 \in (\frac{2}{9}(1-3\epsilon)k, \frac{1}{6}(2-3\epsilon)k]$, which implies $\delta_1^{(2)} \leq \delta_1^{(1)} \leq \delta_2^{(2)} \leq \delta_2^{(1)}$. Obviously, we have $G'_1(b_1) = 1$ and $G'_2(b_1) = -1$ for $b_1 \in [\delta_1^{(1)}, \delta_2^{(2)}]$. As a result, the interval of $[\delta_1^{(1)}, \delta_2^{(2)}]$ is optimal.
 - (3.4) $z_1 \in (\frac{1}{6}(2-3\epsilon)k, \infty)$, which implies $\delta_1^{(2)} \leq \delta_2^{(2)} \leq \delta_1^{(1)} \leq \delta_2^{(1)}$. Since $b_1^* \in [\delta_2^{(2)}, \delta_1^{(1)}]$ satisfies (12) with $b_1^* - 2z_1 \leq 0$ and $(b_1^* - k) + 2(z_1 + \epsilon k) \geq 0$, we have $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.
 - If $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, then $\delta_1^{(2)} \leq 0 \leq \delta_1^{(1)}$. We discuss the following two subcases:
 - (3.5) $z_1 \in (0, \frac{2}{9}(1-\frac{3}{2}\epsilon)k]$, which implies $\delta_1^{(2)} \leq \delta_1^{(1)} \leq \delta_2^{(1)} \leq \delta_2^{(2)}$. Then, $G'_1(b_1) + G'_2(b_1) = 0$ is valid for any $b_1 \in [\delta_1^{(1)}, \delta_2^{(1)}]$. Therefore, the interval of $[\delta_1^{(1)}, \delta_2^{(1)}]$ is optimal.
 - (3.6) $z_1 \in (\frac{2}{9}(1-\frac{3}{2}\epsilon)k, \frac{1}{6}(2-3\epsilon)k]$, which implies $\delta_1^{(2)} \leq \delta_1^{(1)} \leq \delta_2^{(2)} \leq \delta_2^{(1)}$. Then, we have $G'_1(b_1) + G'_2(b_1) = 0$ for any $b_1 \in [\delta_1^{(1)}, \delta_2^{(2)}]$. Therefore, the interval of $[\delta_1^{(1)}, \delta_2^{(2)}]$ is optimal.
 - (3.7) $z_1 \in (\frac{1}{6}(2-3\epsilon)k, \infty]$, leading to $\delta_1^{(2)} \leq \delta_2^{(2)} \leq \delta_1^{(1)}$. Note that $G'_1(\delta_2^{(2)}) \leq G'_1(\delta_1^{(1)}) = 1$ and $G'_2(\delta_1^{(1)}) \geq G'_2(\delta_2^{(2)}) = -1$. Consequently,

$$G'_1(\delta_2^{(1)}) + G'_2(\delta_2^{(1)}) \leq 0 \leq G'_1(\delta_1^{(2)}) + G'_2(\delta_1^{(2)}),$$

Therefore, $b_1^* \in [\delta_2^{(1)}, \delta_1^{(2)}]$ and thus satisfies (12). As $z_1 + \epsilon k \geq z_1 \geq 0$, we derive that $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.

- If $\epsilon \in (\frac{2}{3}, 1)$, then $\delta_1^{(2)} \leq \delta_2^{(2)} \leq \delta_1^{(1)}$, which is the same as in subcase (3.7). Consequently, $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.

Combining the above three cases of z_1 , the optimal solution $b_1^*(z_1)$ to the interpolating spline problem takes the explicit form of (9), (10) and (11), when $\epsilon \in (0, \frac{1}{3})$, $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$ and $\epsilon \in (\frac{2}{3}, 1)$, respectively.

From Theorem 1, we see that three cases of $\epsilon \in (0, 1)$ correspond to different expressions of interpolating spline. In each case, the splines fit can be calculated accordingly.

3.2 Spline Fit Problem

Recall the cubic L^1 spline fitting problem as follows:

$$\min_{z_1} \int_{-1}^1 |z(x) - f(x)| dx,$$

where $z(x)$ is the interpolating spline defined by z_1 and $b_1^*(z_1)$. In the remaining, let $h(x) = z(x) - f(x)$:

$$h(x) = \begin{cases} (-b_1^* + 3z_1)(x+1)^2 + (b_1^* - 2z_1)(x+1)^3, & x \in [-1, 0], \\ z_1 + b_1^*x + [-2b_1^* + (2-3\epsilon)k - 3z_1]x^2 \\ \quad + [b_1^* - (1-2\epsilon)k + 2z_1]x^3, & x \in [0, \epsilon], \\ z_1 + \epsilon k + (b_1^* - k)x + [-2b_1^* + (2-3\epsilon)k - 3z_1]x^2 \\ \quad + [b_1^* - (1-2\epsilon)k + 2z_1]x^3, & x \in [\epsilon, 1]. \end{cases}$$

Note that $b_1^*(z_1)$ takes different forms when $\epsilon \in (0, \frac{1}{3})$, $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$ and $\epsilon \in (\frac{2}{3}, 1)$, respectively. In each case, given the expression of b_1^* , we adopt an analytic approach to find z_1^* that minimizes the spline fitting functional $F(z_1) = \int_{-1}^1 |h(x)| dx$.

Lemma 2 When $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$ and $k > 0$, given $z_0 = 0$, $b_0 = 0$, $z_2 = (1 - \epsilon)k$ and $b_2 = k$, $z_1^* = 0$ minimizes the spline fitting functional $F(z_1) = \int_{-1}^1 |h(x)| dx$.

Proof When $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, the first derivative $b_1^*(z_1)$ takes the explicit expression of (10). Then there are five intervals of z_1 to investigate. Given the value of z_1 , the expression of $F(z_1)$ depends on the sign of $h(x)$ in intervals $[-1, 0]$, $[0, \epsilon]$ and $[\epsilon, 1]$ of x .

- $z_1 \in (-\infty, -\epsilon k)$: $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.
 - $x \in [-1, 0]$:
Since $b_1^* > 2z_1 > 0$, we can derive that $h(x) = (b_1^* - 2z_1)(x + \frac{z_1}{b_1^* - 2z_1})(x + 1)^2 \leq 0$.
 - $x \in [\epsilon, 1]$:

Since $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$ satisfies $\frac{b_1^*}{z_1} = -\frac{b_1^* - k}{z_1 + \epsilon k}$, we have

$$h(x) = (z_1 + \epsilon k) \left(2 - \frac{b_1^*}{z_1} \right) \left(x - \frac{z_1}{b_1^* - 2z_1} \right) (x - 1)^2.$$

Following $z_1 \leq -\epsilon k$ and $b_1 \geq 2z_1$, we obtain $h(x) \leq 0$.
– $x \in [0, \epsilon]$:

Note that $h'(x) = b_1 + 3(z_1 + \epsilon k)(2 - \frac{b_1^*}{z_1})(x - \frac{4b_1^* - 6z_1}{3b_1^* - 6z_1})x \geq 0$ holds for any $x \in [0, \epsilon]$. Since $h(0) \leq 0$ and $h(\epsilon) \leq 0$, we have $h(x) \leq 0$.

Subsequently, $|h(x)| = -h(x)$ holds for any $x \in [-1, 1]$. Then, $F(z_1)$ can be calculated by

$$\begin{aligned} F(z_1) &= -\int_{-1}^0 h(x)dx - \int_0^\epsilon h(x)dx - \int_\epsilon^1 h(x)dx \\ &= \frac{1}{12}(b_1^* - 6z_1) - \frac{1}{12}(b_1^* + 6z_1) + C \\ &= -z_1 + C, \end{aligned}$$

where C is a constant. Therefore, $F'(z_1) = -1 < 0$ for $z_1 \in (-\infty, -\epsilon k)$.

- $z_1 \in (-\epsilon k, \frac{1}{4}(1 - 3\epsilon)k)$: $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$.
– $x \in [-1, 0]$:

Similar as the above case, $b_1^* \geq 0 \geq 2z_1$ leads to $h(x) = (b_1^* - 2z_1)(x + \frac{z_1}{b_1^* - 2z_1})(x + 1)^2 \leq 0$.

- $x \in [\epsilon, 1]$:

Since $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$ satisfies $\frac{b_1^*}{z_1} - 4 = \frac{b_1^* - k}{z_1 + \epsilon k}$, we have

$$h(x) = (z_1 + \epsilon k) \left[\left(\frac{b_1^*}{z_1} - 2 \right) x + 1 \right] (x - 1)^2,$$

- where $(\frac{b_1^*}{z_1} - 2) = 4\frac{z_1 + \frac{1}{4}(3\epsilon - 1)k}{\epsilon k} - 1 \in [-2 - \frac{1}{\epsilon}, -1]$. Therefore, $h(x) \leq 0$.
– $x \in [0, \epsilon]$:

Note that $h'(x) = b_1 + 3(z_1 + \epsilon k)(\frac{b_1^*}{z_1} - 2)(x - \frac{4b_1^* - 10z_1}{3b_1^* - 6z_1})x \geq 0$ holds for any $x \in [0, \epsilon]$. Since $h(0) \leq 0$ and $h(\epsilon) \leq 0$, we obtain $h(x) \leq 0$.

In this case, $F(z_1) = -\int_{-1}^1 h(x)dx$ and $F'(z_1) = -1 < 0$.

- $z_1 \in (\frac{1}{4}(1 - 3\epsilon)k, 0]$: $b_1^* = z_1$.
– $x \in [-1, 0]$: $h(x) = -z_1(x - 1)(x + 1)^2 \leq 0$.
– $x \in [\epsilon, 1]$:

$$\begin{aligned} h(x) &= \{[3z_1 - (1 - 2\epsilon)k]x + z_1 + \epsilon k\} (x - 1)^2 \\ &= \{[4z_1 - (1 - 3\epsilon)k]x + (z_1 + \epsilon k)(1 - x)\} (x - 1)^2 \end{aligned}$$

- Note that $4z_1 - (1 - 3\epsilon)k \geq 0$ and $z_1 + \epsilon k \geq 0$, hence, $h(x) \geq 0$ for any $x \in [\epsilon, 1]$.
– $x \in [0, \epsilon]$:

$$h(x) = z_1 + z_1x + [-5z_1 + (2 - 3\epsilon)k]x^2 + [3z_1 - (1 - 2\epsilon)k]x^3.$$

Since $h'(x) = z_1 + 2[(2 - 3\epsilon)k - 5z_1]x + 3[3z_1 - (1 - 2\epsilon)k]x^2$, we can calculate $h'(0) = z_1 \leq 0$ and $h'(\epsilon) = (9\epsilon - 1)(\epsilon - 1)z_1 + (4\epsilon - 9\epsilon^2 + 6\epsilon^3)k \geq 0$.

Meanwhile, $h(0) = z_1 \leq 0$ and $h(\epsilon) \geq 0$ imply that there is a unique $x^* \in [0, \epsilon]$ with $h(x^*) = 0$.

In this case,

$$F(z_1) = - \int_{-1}^0 h(x) dx - \int_0^{x^*} h(x) dx + \int_{x^*}^{\epsilon} h(x) dx + \int_{\epsilon}^1 h(x) dx,$$

and

$$\begin{aligned} F'(z_1) &= - \int_{-1}^0 \frac{dh(x)}{dz_1} dx - \int_0^{x^*} \frac{dh(x)}{dz_1} dx - \int_{x^*}^{\epsilon} \frac{dh(x)}{dz_1} dx + \int_{\epsilon}^1 \frac{dh(x)}{dz_1} dx \\ &= - \int_{-1}^{\epsilon} \frac{dh(x)}{dz_1} dx + \int_{\epsilon}^1 \frac{dh(x)}{dz_1} dx \\ &= \frac{1}{6} - 2\epsilon - \epsilon^2 + \frac{10}{3}\epsilon^3 - \frac{3}{2}\epsilon^4 \\ &< 0. \end{aligned}$$

- $z_1 \in (0, \frac{1}{6}(2 - 3\epsilon)k]$: $b_1^* \in [\frac{3}{2}z_1, \min\{3z_1, (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1\}]$.
– $x \in [-1, 0]$:

Since $b_1^* \leq 3z_1$, we have $h(x) = b_1^*[(x+1)^3 - (x+1)^2] + z_1[3(x+1)^2 - 2(x+1)^3] \geq z_1(x+1)^3 \geq 0$.

- $x \in [\epsilon, 1]$: Since $b_1^* \geq \frac{3}{2}z_1$,

$$\begin{aligned} h(x) &= \{[b_1^* - (1 - 2\epsilon)k + 2z_1]x + (z_1 + \epsilon k)\}(x-1)^2 \\ &\geq \left\{ \left[\frac{7}{2}z_1 - (1 - 2\epsilon)k \right] x + z_1 + \epsilon k \right\} (x-1)^2 \\ &= \left\{ \left[\frac{9}{2}z_1 - (1 - 3\epsilon)k \right] x + (z_1 + \epsilon k)(1-x) \right\} (x-1)^2 \end{aligned}$$

Note that $\frac{9}{2}z_1 - (1 - 3\epsilon)k \geq 0$ and $z_1 + \epsilon k \geq 0$, hence, $h(x) \geq 0$ for any $x \in [\epsilon, 1]$.

- $x \in [0, \epsilon]$:

Because $h'(x) = \frac{3}{2}z_1 + 2[(2 - 3\epsilon)k - 6z_1]x + 3[\frac{7}{2}z_1 - (1 - 2\epsilon)k]x^2$ with $h'(0) = \frac{3}{2} > 0$ and $h''(x) = 2[-6z_1 + (2 - 3\epsilon)k] + 3[7z_1 - 2(1 - 2\epsilon)k]x \geq 0$ on $[0, 1]$, we have $h'(x) \geq 0$ for any $x \in [0, \epsilon]$. Note that $h(0) \geq 0$ and $h(\epsilon) \geq 0$, consequently, $h(x) \geq 0$.

In this case, $F(z_1) = \int_{-1}^1 h(x) dx$ and $F'(z_1) = 1 > 0$.

- $z_1 \in (\frac{1}{6}(2 - 3\epsilon)k, \infty)$: $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.
– $x \in [-1, 0]$:

Because $b_1^* < 2z_1 < 3z_1$, we have $h(x) = (b_1^* - 2z_1)(x + \frac{z_1}{b_1^* - 2z_1})(x+1)^2 \geq 0$.

– $x \in [\epsilon, 1]$:

Since $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$ satisfies $\frac{b_1^*}{z_1} = -\frac{b_1^* - k}{z_1 + \epsilon k}$,

$$h(x) = (z_1 + \epsilon k) \left(2 - \frac{b_1^*}{z_1} \right) \left(x - \frac{z_1}{b_1^* - 2z_1} \right) (x - 1)^2 \geq 0.$$

– $x \in [0, \epsilon]$:

Note that $h'(x) = b_1^* + 3(z_1 + \epsilon k)(2 - \frac{b_1^*}{z_1})(x - \frac{4b_1^* - 6z_1}{3b_1^* - 6z_1})x$ with $h'(0) = b_1^* > 0$ and $h''(x) \geq 0$, hence, $h'(x) \geq 0$ holds for any $x \in [0, \epsilon]$. Because $h(0) \geq 0$ and $h(\epsilon) \geq 0$, we obtain $h(x) \geq 0$.

In this case, $F(z_1) = \int_{-1}^1 h(x)dx$ and $F'(z_1) = 1 > 0$.

From the above discussions, we have shown that $F(z_1)$ is strictly convex in terms of $z_1 \in \mathbb{R}$ with $F'(0) = 0$. Therefore, the L^1 spline fit is given by $z_1^* = 0$ with $b_1^*(z_1^*) = 0$.

Similarly, we can derive the spline fit for $f(x)$ with $\epsilon \in (0, \frac{1}{3})$. The result is presented in the following lemma.

Lemma 3 When $\epsilon \in (0, \frac{1}{3})$ and $k > 0$, given $z_0 = 0, b_0 = 0, z_2 = (1 - \epsilon)k$ and $b_2 = k, z_1^* = 0$ minimizes the spline fitting functional $F(z_1) = \int_{-1}^1 |h(x)|dx$.

Proof We adopt the same analytic approach as in the proof of Theorem 1 to show the convexity of $F(z_1)$. When $\epsilon \in (0, \frac{1}{3})$, the first derivative $b_1^*(z_1)$ takes the explicit expression of (9). Then there are six intervals of z_1 to investigate.

- $z_1 \in (-\infty, -\epsilon k)$: $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.

Following the discussion in the same interval of z_1 when $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, we have $h(x) \leq 0$ for $x \in [-1, 1]$ and $F'(z_1) = -1 < 0$ for $z_1 \in (-\infty, -\epsilon k]$.

- $z_1 \in (-\epsilon k, 0)$: $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$.
– $x \in [-1, 0]$:

Since $b_1^* \geq 0 \geq 2z_1$, we have $h(x) = (b_1^* - 2z_1)(x + \frac{z_1}{b_1^* - 2z_1})(x + 1)^2 \leq 0$.

– $x \in [\epsilon, 1]$:

Note that $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$ satisfies $-\frac{2z_1}{b_1^* - 2z_1} = -\frac{2(z_1 + \epsilon k)}{(b_1^* - k) + 2(z_1 + \epsilon k)}$, then

$$h(x) = (z_1 + \epsilon k) \left[\frac{4z_1 + (2\epsilon - 1)k}{\epsilon k} x + 1 \right] (x - 1)^2.$$

As $z_1 \in (-\epsilon k, 0)$, it can be derived that $\frac{4z_1 + (2\epsilon - 1)k}{\epsilon k} \in (-2 - \frac{1}{\epsilon}, 2 - \frac{1}{\epsilon})$. The sign of $h(x)$ depends on the value of $z_1 \in (-\epsilon k, 0)$. We discuss the following two subcases:

- (1) $z_1 \in (-\epsilon k, -\frac{1}{2}\epsilon k]$. Then, $\frac{4z_1+(2\epsilon-1)k}{\epsilon k}(x-1)+1 < 0$ for any $x \in [\epsilon, 1]$, based on which we have $h(x) \leq 0$.
 - (2) $z_1 \in (-\frac{1}{2}\epsilon k, 0)$. Denoting $\tilde{x} = -\frac{\epsilon k}{4z_1+(2\epsilon-1)k} \in [\epsilon, 1]$, it can be verified that $h(x) \geq 0$ on $[\epsilon, \tilde{x}]$ and $h(x) \leq 0$ on $[\tilde{x}, 1]$.
- $x \in [0, \epsilon]$:

Follow the above two subclasses. If $z_1 \in (\epsilon k, -\frac{1}{2}\epsilon k]$, then $h(x) \leq 0$ for any $x \in [0, \epsilon]$. If $z_1 \in (-\frac{1}{2}\epsilon k, 0]$, then there is a unique $\bar{x} \in [0, \epsilon]$ such that $h(x) \leq 0$ on $[0, \bar{x}]$ and $h(x) \geq 0$ on $[\bar{x}, \epsilon]$. Therefore, $F(z_1)$ is calculated for $z_1 \in (\epsilon k, -\frac{1}{2}\epsilon k]$ and $z_1 \in (-\frac{1}{2}\epsilon k, 0]$, respectively.

If $z_1 \in (\epsilon k, -\frac{1}{2}\epsilon k]$, then

$$F(z_1) = -\int_{-1}^1 h(x)dx \text{ and } F'(z_1) = -1 < 0.$$

If $z_1 \in (-\frac{1}{2}\epsilon k, 0)$, then

$$F(z_1) = -\int_{-1}^0 h(x)dx - \int_0^{\tilde{x}} h(x)dx + \int_{\tilde{x}}^{\epsilon} h(x)dx + \int_{\epsilon}^1 h(x)dx - \int_{\tilde{x}}^1 h(x)dx,$$

and

$$\begin{aligned} F'(z_1) &= -\int_{-1}^{\epsilon} \frac{dh(x)}{dz_1} dx + \int_{\epsilon}^1 \frac{dh(x)}{dz_1} dx \\ &= \left(\frac{1}{6} - \epsilon^2 + \frac{4}{3}\epsilon^3 - \frac{1}{2}\epsilon^4 \right) \frac{db_1^*}{dz_1} - 2\epsilon + 2\epsilon^3 - \epsilon^4 \\ &= \left(\frac{1}{6} - \epsilon^2 + \frac{4}{3}\epsilon^3 - \frac{1}{2}\epsilon^4 \right) \left[\left(\frac{8z_1}{k} - 1 \right) \frac{1}{\epsilon} + 4 \right] - 2\epsilon + 2\epsilon^3 - \epsilon^4. \end{aligned}$$

Since $\epsilon \in (0, \frac{1}{3})$ and $z_1 \in (-\frac{1}{2}\epsilon k, 0)$, we find that $F'(z_1) < 0$ for all $z_1 \in (-\frac{1}{2}\epsilon k, 0)$.

- $z_1 \in (0, \frac{1}{6}(1-3\epsilon)k)$: $b_1^* = \frac{kz_1}{2z_1+\epsilon k}$.
– $x \in [-1, 0]$:

Note that $h(x) = (b_1^* - 2z_1)(x + \frac{z_1}{b_1^* - 2z_1})(x+1)^2$. Since $b_1^* > 3z_1 \geq 2z_1 \geq 0$ and $\bar{x} = -\frac{z_1}{b_1^* - 2z_1} \in [-1, 0]$, we have

$$h(x) \begin{cases} \leq 0, & x \in [-1, \bar{x}], \\ \geq 0, & x \in [\bar{x}, 0]. \end{cases}$$

– $x \in [\epsilon, 1]$:

Consider $h(x) = (z_1 + \epsilon k)(2 - \frac{b_1^*}{z_1})(x - \frac{z_1}{b_1^* - 2z_1})(x - 1)^2$ and let $\tilde{x} = \frac{z_1}{b_1^* - 2z_1}$. Since $z_1 \in (0, \frac{1}{6}(1 - 3\epsilon)k)$, we derive that $\tilde{x} \in [\frac{\epsilon}{2\epsilon - 1}, 1]$ and then $\epsilon < \tilde{x} \leq 1$ holds. Consequently,

$$h(x) \begin{cases} \geq 0, & x \in [\epsilon, \tilde{x}], \\ \leq 0, & x \in [\tilde{x}, 1]. \end{cases}$$

– $x \in [0, \epsilon]$:

$h'(x) \geq 0$, $h(0) \geq 0$ and $h(\epsilon) \geq 0$ lead to $h(x) \geq 0$ for $x \in [0, \epsilon]$.
Combining the above three intervals, we can calculate $F(z_1)$ as

$$F(z_1) = - \int_{-1}^{\tilde{x}} h(x) dx + \int_{\tilde{x}}^0 h(x) dx + \int_0^{\epsilon} h(x) dx + \int_{\epsilon}^{\tilde{x}} h(x) dx - \int_{\tilde{x}}^1 h(x) dx,$$

and

$$F'(z_1) = - \int_{-1}^0 \frac{dh(x)}{dz_1} dx + \int_0^1 \frac{dh(x)}{dz_1} dx = \frac{1}{6} \frac{db_1^*}{dz_1} = \frac{\epsilon k^2}{6(2z_1 + \epsilon k)^2} > 0.$$

- $z_1 \in [\frac{1}{6}(1 - 3\epsilon)k, \frac{2}{9}(1 - 3\epsilon)k)$: $b_1^* \in [(1 - 3\epsilon)k - 3z_1, 3z_1]$.
Following $(1 - 3\epsilon)k - 3z_1 \leq b_1^* \leq 3z_1$, we have

– $x \in [-1, 0]$:

$$h(x) = b_1^*[(x + 1)^3 - (x + 1)^2] + z_1[-2(x + 1)^3 + 3(x + 1)^2] \geq z_1(x + 1)^3 \geq 0.$$

– $x \in [\epsilon, 1]$:

$$\begin{aligned} h(x) &= \{[b_1^* - (1 - 2\epsilon)k + 2z_1]x + (z_1 + \epsilon k)\}(x - 1)^2 \\ &\geq \{[(1 - 3\epsilon)k - 3z_1 - (1 - 2\epsilon)k + 2z_1]x + (z_1 + \epsilon k)\}(x - 1)^2 \\ &= -(z_1 + \epsilon k)(x - 1)^3 \geq 0. \end{aligned}$$

– $x \in [0, \epsilon]$:

Since $h'(x) \geq 0$, $h(0) \geq 0$ and $h(\epsilon) \geq 0$, we have $h(x) \geq 0$ for $x \in [0, \epsilon]$.
Combining the above three intervals, we see that $h(x) \geq 0$ for $x \in [-1, 1]$ and then $F(z_1) = \int_{-1}^1 h(x) dx$ with $F'(z_1) = 1 > 0$.

- $z_1 \in [\frac{2}{9}(1 - 3\epsilon)k, \frac{1}{6}(2 - 3\epsilon)k)$: $b_1^* \in [\frac{3}{2}z_1, (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1]$.
In this case, $\frac{3}{2}z_1 \leq b_1^* < 2z_1$.

– $x \in [-1, 0]$:

$$h(x) = b_1^*((x + 1)^3 - (x + 1)^2) + z_1(-2(x + 1)^3 + 3(x + 1)^2) \geq z_1(x + 1)^2 \geq 0.$$

– $x \in [\epsilon, 1]$:

$$\begin{aligned} h(x) &= \{[b_1^* - (1 - 2\epsilon)k + 2z_1]x + (z_1 + \epsilon k)\}(x - 1)^2 \\ &\geq \left\{ \left[\frac{7}{2}z_1 - (1 - 2\epsilon)k \right]x + z_1 + \epsilon k \right\} (x - 1)^2 \\ &= \left\{ \left[\frac{9}{2}z_1 - (1 - 3\epsilon)k \right]x + (z_1 + \epsilon k)(2 - x) \right\} (x - 1)^2 \geq 0. \end{aligned}$$

– $x \in [0, \epsilon]$:

Since $h'(x) \geq 0$, $h(0) \geq 0$ and $h(\epsilon) \geq 0$, we have $h(x) \geq 0$ for $x \in [0, \epsilon]$. Combining the above three intervals, we derive that $h(x) \geq 0$ for $x \in [-1, 1]$ and then $F(z_1) = \int_{-1}^1 h(x)dx$ with $F'(z_1) = 1 > 0$.

- $z_1 \in [\frac{1}{6}(2 - 3\epsilon)k, \infty)$: $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.

Following the discussion in the same interval of z_1 when $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, $h(x) \geq 0$ for $x \in [-1, 1]$ and $F'(z_1) = 1 > 0$ for $z_1 \in [\frac{1}{6}(2 - 3\epsilon)k, \infty)$.

From the above discussions, we have shown that $F(z_1)$ is strictly convex in terms of $z_1 \in \mathbb{R}$ with $F'(0) = 0$. Therefore, the L^1 spline fit is given by $z_1^* = 0$ with $b_1^*(z_1^*) = 0$.

Remark 2 When $\epsilon = 0$, which is a critical value of ϵ , $z_1^* \in [0, \frac{1}{6}k]$ solves the spline fitting problem with the spline interpolating problem solved by

$$b_1^*(z_1) = \begin{cases} \frac{k}{2}, & \text{if } z_1 \in (-\infty, 0) \cup (0, \frac{1}{6}k) \cup [\frac{1}{3}k, \infty), \\ b_1 \in [0, k], & \text{if } z_1 = 0, \\ b_1 \in [k - 3z_1, 3z_1], & \text{if } z_1 \in [\frac{1}{6}k, \frac{2}{9}k], \\ b_1 \in [\frac{3}{2}z_1, k - \frac{3}{2}z_1], & \text{if } z_1 \in [\frac{2}{9}k, \frac{1}{3}k]. \end{cases}$$

The following lemma shows the result when $\epsilon \in (\frac{2}{3}, 1)$.

Lemma 4 When $\epsilon \in (\frac{2}{3}, 1)$ and $k > 0$, given $z_0 = 0$, $b_0 = 0$, $z_2 = (1 - \epsilon)k$ and $b_2 = k$, $z_1^* = 0$ minimizes the spline fitting functional $F(z_1) = \int_{-1}^1 |h(x)|dx$.

Proof Similarly, we prove that $F(z_1)$ is a strictly convex function with $F'(z_1) = 0$. Based on the explicit expression (11) of $b_1^*(z_1)$, there are five intervals of z_1 to investigate.

- $z_1 \in (-\infty, -\epsilon k)$: $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.

Following the discussion in same interval of z_1 when $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, we have $h(x) \leq 0$ for $x \in [0, 2]$ and $F'(z_1) = -1 < 0$.

- $z_1 \in (-\epsilon k, \frac{1}{4}(1 - 3\epsilon)k]$: $b_1^* = \frac{4z_1^2 + (4\epsilon - 1)kz_1}{\epsilon k}$.

Following the discussion in same interval of z_1 when $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, we have $h(x) \leq 0$ for $x \in [0, 2]$ and $F'(z_1) = -1 < 0$.

- $z_1 \in (\frac{1}{4}(1 - 3\epsilon)k, \frac{1}{5}(2 - 3\epsilon)k]$: $b_1^* = z_1$.

Following the discussion in same interval of z_1 when $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, we have $F'(z_1) = \frac{1}{6} - 2\epsilon - \epsilon^2 + \frac{10}{3}\epsilon^3 - \frac{3}{2}\epsilon^4 < 0$.

- $z_1 \in (\frac{1}{5}(2 - 3\epsilon)k, 0)$: b_1^* solves $\left(\frac{2z_1}{b_1^* - 2z_1}\right)^2 + \left(\frac{2(z_1 + \epsilon k)}{(b_1^* - k) + 2(z_1 + \epsilon k)}\right)^2 = 10$.

In this case, $b_1^* \in [\max\{\frac{3}{2}z_1, (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1\}, z_1]$.

– $x \in [-1, 0]$:

Because $0 > b_1^* > 2z_1$, we have $h(x) = (b_1^* - 2z_1)(x + \frac{z_1}{b_1^* - 2z_1})(x + 1)^2 \leq 0$.

– $x \in [\epsilon, 1]$:

$$h(x) = \{[b_1^* - (1 - 2\epsilon)k + 2z_1]x + (z_1 + \epsilon k)\}(x - 1)^2$$

Since $b_1^* \geq (1 - \frac{3}{2}\epsilon)k - \frac{3}{2}z_1$ and $b_1^* \geq \frac{3}{2}z_1$, it can be derived that $[b_1^* - (1 - 2\epsilon)k + 2z_1] \geq 0$ for $z_1 \in (\frac{1}{5}(2 - 3\epsilon)k, 0)$. Subsequently, $h(x) \geq 0$.

– $x \in [0, \epsilon]$:

Note that $h(0) \leq 0$ and $h(\epsilon) \geq 0$. Since $h'(x) \geq 0$ on $[0, \epsilon]$, there exists $x^* \in [0, \epsilon]$ such that $h(x) \leq 0$ for $x \in [0, x^*]$ and $h(x) \geq 0$ for $x \in [x^*, \epsilon]$.

Consequently,

$$F(z_1) = - \int_{-1}^0 h(x)dx - \int_0^{x^*} h(x)dx + \int_{x^*}^{\epsilon} h(x)dx + \int_{\epsilon}^1 h(x)dx,$$

and

$$F'(z_1) = \frac{1}{6} - 2\epsilon - \epsilon^2 + \frac{10}{3}\epsilon^3 - \frac{3}{2}\epsilon^4 < 0.$$

- $z_1 \in [0, \infty)$: $b_1^* = \frac{kz_1}{2z_1 + \epsilon k}$.

Following the discussion in same interval of z_1 when $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$, we have $F(z_1) = \int_{-1}^1 h(x)dx$ and $F'(z_1) = 1 > 0$.

From the above discussions, we have shown that $F(z_1)$ is strictly convex in terms of $z_1 \in \mathbb{R}$ with $F'(0) = 0$. Therefore, the spline fit of $f(x)$ is defined by $z_1^* = 0$ with $b_1^*(z_1^*) = 0$.

Based on the above lemmas, we conclude the result of 3-node-window case in the following theorem.

Theorem 5 *In a 3-node window, given $z_0 = 0$, $b_0 = 0$, $z_2 = (1 - \epsilon)k$ and $b_2 = k$ with $\epsilon \in (0, 1)$ and $k > 0$, the second-derivative-based spline fit for the convex corner $f(x)$ is expressed as*

Table 1 The spline fit in a 3-node window

	$\epsilon \in (0, \frac{1}{3})$	$\epsilon \in [\frac{1}{3}, \frac{2}{3}]$	$\epsilon \in (\frac{2}{3}, 1)$
$z(0)$	0	0	0
$z'(0)$	0	0	0
$z''(-1)$	0	0	0
$z''(0^-)$	0	0	0
$z''(0^+)$	≥ 0	≥ 0	< 0
$z''(1)$	< 0	≥ 0	≥ 0
$z(x)$	Nonconvex	Convex	Nonconvex

$$z(x) = \begin{cases} 0, & \text{if } x \in [-1, 0], \\ (2 - 3\epsilon)kx^2 + (2\epsilon - 1)kx^3, & \text{if } x \in [0, 1]. \end{cases} \quad (13)$$

Moreover, the spline fit $z(x)$ is convex in the window if and only if $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$.

Proof Following Lemma 2, Lemma 3 and Lemma 4, the spline fitting functional with any given $\epsilon \in (0, 1)$ is minimized by $z_1^* = 0$ with $b_1^*(z_1) = 0$. Plugging the value of z_1^* and $b^*(z_1)$ into $z(x)$ of form (3), we obtain the desired expression (13).

Note that the spline fit $z(x)$ is linear in the first interval, it reduces to check the convexity of $z(x)$ in the second interval. Base on (13), we have

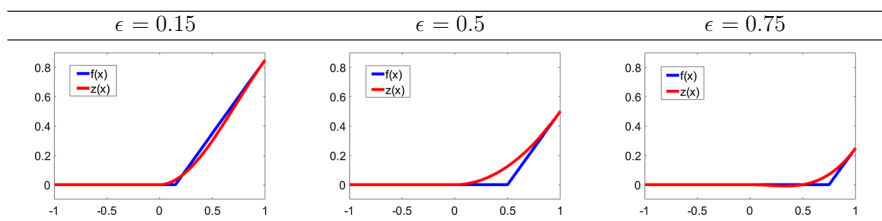
$$z''((-1)^+) = 6\left(\frac{2}{3} - \epsilon\right); \quad z''(0^-) = 6\left(\epsilon - \frac{1}{3}\right).$$

Subsequently, both $z''((-1)^+)$ and $z''(0^-)$ are nonnegative if and only if $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$.

Table 1 lists the first derivative, second derivative and convexity of the spline fit $z(x)$ for the convex corner $f(x)$ in a 3-node window. From Table 1, we see that the first derivative at the middle spline node is 0 in all three cases of ϵ . Therefore, the linear shape of the longer line segment on the left is always preserved by the spline fit in a 3-node window. However, the spline fit can preserve the convex shape if and only if the horizontal length difference ϵ of two line segments is within the middle third of interval's length.

Table 2 shows the graphs of spline fit in three different cases of $\epsilon \in (0, \frac{1}{3})$, $\epsilon \in [\frac{1}{3}, \frac{2}{3}]$ and $\epsilon \in (\frac{2}{3}, 1)$, respectively. When ϵ is relatively small, the spline fit is concave at the end of second interval; When ϵ is relatively large, the spline fit is concave at the beginning of second interval. Intuitively, a convex interpolating spline in the window must be positioned above the convex corner. When ϵ is relatively small or large, the absolute gap between the convex corner and a convex spline may become large around the break point, and thus convex interpolating splines will not be the spline fit that minimizes the L^1 error.

Table 2 The graph of spline fit in a 3-node window, $k = 1$



4 Second-Derivative-Based Spline Fit in a 5-Node Window

A window of five nodes has been shown of appropriate size for both efficiency and accuracy in locally calculating the interpolating spline and spline fits. For the convex corner in the form of $f(x)$, we compute the spline fit in a 5-node window without boundary conditions where $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$. The similar analysis as in Sect. 3 is extremely complex for the 5-node window case. Instead, we adopt the following numerical computational procedure:

Step 1: Set a precision increment of $0 < \Delta\epsilon < 1$. Let $\epsilon = 0$.

Step 2: Solve the spline fit problem numerically.

- If $z_0 = z_1 = z_2 = 0$ and $b_0 = b_1 = b_2 = 0$, set $LL(\epsilon) = 1$. Otherwise, set $LL(\epsilon) = 0$.
- Compute $z''(x_1^-)$, $z''(x_2^-)$, $z''(x_3^-)$, $z''(x_4^-)$, $z''(x_0^+)$, $z''(x_1^+)$, $z''(x_2^+)$, and $z''(x_3^+)$. If $z''(x_1^-) \geq 0$, $z''(x_2^-) \geq 0$, $z''(x_3^-) \geq 0$, $z''(x_4^-) \geq 0$, $z''(x_0^+) \geq 0$, $z''(x_1^+) \geq 0$, $z''(x_2^+) \geq 0$, and $z''(x_3^+) \geq 0$, set $Conv(\epsilon) = 1$. Otherwise, set $Conv(\epsilon) = 0$.
- If $z_3 = k - \epsilon$, $z_4 = 2k - \epsilon$ and $b_3 = b_4 = k$, set $LR(\epsilon) = 1$. Otherwise, set $LR(\epsilon) = 0$.

let $\epsilon = \epsilon + \Delta\epsilon$ and repeat Step 2.

Step 3: If $\epsilon = 1$. Stop.

In Step 2, we need to solve the interpolating spline problem for $b^*(z)$. We use the Lagrangian multiplier primal affine method as in [11] and [13]. This algorithm is satisfyingly accurate for our purpose of comparing the metrics. We use the MATLAB function `fmincon` to numerically minimize the L^1 fitting functional, which is in general not analytically expressed. We set $k = 1$ and $\Delta\epsilon = 0.01$. The starting point is set as $z = (0, 0, 0, 1 - \epsilon, 2 - \epsilon)$. The criterion of $z_0 = z_1 = z_2 = 0$ is that $|z_0| + |z_1| + |z_2| < 10^{-4}$. The criterion of $b_0 = b_1 = b_2 = 0$ is that $|b_0| + |b_1| + |b_2| < 10^{-4}$. The criterion of $z''(x_i^-) \geq 0$ is $z''(x_i^-) > -10^{-4}$ for $i = 1, 2, 3, 4$. The criterion of $z''(x_i^+) \geq 0$ is $z''(x_i^+) > -10^{-4}$ for $i = 0, 1, 2, 3$. The criterion of $z_3 = k - \epsilon$, $z_4 = 2k - \epsilon$ is that $|z_3 - (k - \epsilon)| + |z_4 - (2k - \epsilon)| < 10^{-4}$. The criterion of $b_3 = b_4 = k$ is that $|b_3 - k| + |b_4 - k| < 10^{-4}$. The outputs $Conv(\epsilon) = 1$, $LL(\epsilon) = 1$ and $LR(\epsilon) = 1$ indicate the spline fit to be convex in all four intervals, linear in the first two intervals and linear in the fourth interval, respectively.

We list the computational result in the Table 3. From Table 3, we see that the range of ϵ that guarantees the spline fit preserving the linear shape in the first, second and

Table 3 L^1 cubic spline fit in a 5-node window

	Convexity (all four intervals)	Linearity (intervals 1 and 2)	Linearity (interval 4)
ϵ	[0.34, 0.67]	[0.04, 0.72]	[0.04, 0.72]

fourth intervals of 5-node window is [0.04, 0.72], smaller than (0, 1) in the 3-node window. This suggests that the 5-node window contains more information about the geometric shape to be approximated.

Meanwhile, the spline fit preserves the convex shape in the 5-node window if and only if $\epsilon \in [0.34, 0.67]$. In this sense, the spline fit has the same capability of preserving the convex shape in 3-node window and 5-node window. The next theorem shows a general property of spline fit in a window of odd-number equally spaced nodes.

Theorem 6 *Let $I \geq 2$ be an even number, $D > 0$ and $k > 0$. Suppose that the spline fit $z(x)$ with equally spaced nodes $x_0 = -\frac{1}{2}D, \dots, x_{I/2} = 0, \dots, x_I = \frac{1}{2}D$, preserves the linear shape of convex corner*

$$f(x) = \begin{cases} 0, & \text{if } x \leq \epsilon, \\ k(x - \epsilon), & \text{if } x \geq \epsilon, \end{cases}$$

in $[x_0, x_{I/2}]$ and $[x_{I/2+1}, x_I]$ as long as $\epsilon \in \mathcal{L} \subseteq (0, D)$. Then, the spline fit $z(x)$ is convex in the window if $\epsilon \in [\frac{1}{3}D, \frac{2}{3}D] \cap \mathcal{L}$.

Proof For $\epsilon \in \mathcal{L}$, the spline fit $z(x)$ is linear in $[x_0, x_{I/2}]$ and $[x_{I/2+1}, x_I]$, respectively. Then, the convexity of $z(x)$ in the window becomes equivalent to that in $[x_{I/2}, x_{I/2+1}]$. Since the function values and first derivatives of $z(x)$ coincide with those of convex corner $f(x)$ at the boundary nodes of $[x_{I/2}, x_{I/2+1}]$, we have

$$z_{I/2} = 0, b_{I/2} = 0; \quad z_{I/2} = k(D - \epsilon), b_{I/2} = k. \quad (14)$$

Plugging (14) into the expression of $z(x)$ in $[x_{I/2}, x_{I/2+1}]$, i.e., $[0, D]$, we obtain

$$z(x) = \frac{k}{D} \left(2 - 3\frac{\epsilon}{D}\right) x^2 + \frac{k}{D^2} \left(2\frac{\epsilon}{D} - 1\right) x^3$$

with $z''(0^+) = \frac{2k}{D} (2 - 3\frac{\epsilon}{D})$ and $z''(D^-) = \frac{2k}{D} (3\frac{\epsilon}{D} - 1)$. Then, $z(x)$ is convex in $[0, D]$ if and only if $z''(0^+), z''(D^-) \geq 0$, i.e., $\epsilon \in [\frac{1}{3}D, \frac{2}{3}D]$. Therefore, if $\epsilon \in [\frac{1}{3}D, \frac{2}{3}D] \cap \mathcal{L}$, then the spline fit $z(x)$ is convex in the window of $[x_0, x_I]$.

Remark 3 The result of Theorem 6 is also applicable to the cubic L^1 spline fit based on the first derivative and function value, respectively.

Note that the result in Table 3 verifies Theorem 6 with $I = 2$, $D = 1$ and $\mathcal{L} = [0.04, 0.72]$. We may conjecture that in a window of 7 or even larger odd-number nodes, the linear-shape-preserving capability of spline fit will decrease as the number

Table 4 Three types of L^1 cubic spline fits in a 5-node window

Spline fits	Convexity (all four intervals)	Linearity (intervals 1 and 2)	Linearity (interval 4)
Second-derivative	$0.34 \leq \epsilon \leq 0.67$	$0.04 \leq \epsilon \leq 0.72$	$0.04 \leq \epsilon \leq 0.72$
First-derivative	$0.34 \leq \epsilon \leq 0.67$	$0.07 \leq \epsilon \leq 0.99$	$0.07 \leq \epsilon \leq 0.68$
Function-value	Never	Never	Never

of spline nodes increases. However, as long as the range of ϵ that guarantees the linear-shape preservation contains $[\frac{1}{3}D, \frac{2}{3}D]$, the corner's convex shape can be preserved by the spline fit when $\epsilon \in [\frac{1}{3}D, \frac{2}{3}D]$.

In real applications, the more spline nodes in the window, the better approximation to the original function is provided by the spline fit. Therefore, while using the cubic L^1 spline fit to approximate a convex corner, choosing the number of nodes placed in the window is a trade-off between the L^1 error of approximation and the shape-preserving performance.

5 Extensions

To fully evaluate the shape-preserving capability of cubic L^1 spline fits for the convex corner, we also need to investigate the first-derivative based and function-value based spline fits. In a 3-node-window, the analytic approach for calculating the second-derivative-based spline fit can be applied to the other two cases, yet may be extremely cumbersome. Alternatively, we use the similar numerical computational procedure in the second-derivative-based case to calculate the other two types of spline fits in a 5-node window. The results compared with that in Table 3 are shown below.

From Table 4, we have the following observations on the spline fits of convex corner:

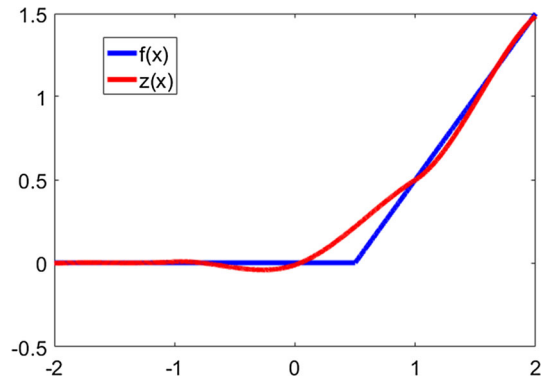
- (i) Between the second-derivative based and first-derivative based spline fits, the latter is better in preserving the linear shape of line segment on the left, while worse in preserving the linear shape of line segment on the right.

Intuitively, this phenomenon may arise from the fact that the first-derivative based spline fits prefer to preserve the linear shape at the beginning, but lose the flexibility of linear-shape preservation in the remaining part of window. Differently, the second-derivative based spline fits have the same flexibility of preserving linear shapes of line segments on the left and on the right.

- (ii) The second-derivative based and first-derivative based spline fits share the same capability of preserving the convex shape.

From Theorem 6, we know that in both cases the range of ϵ for guaranteeing the convex-shape preservation is the intersection of that for linear-shape preservation and $[\frac{1}{3}, \frac{2}{3}]$. Since the range of ϵ for linear-shape preservation contains $[\frac{1}{3}, \frac{2}{3}]$ in both cases, it is reasonable that these two types of spline fits share the same capability of preserving the convex shape in the 5-node window.

Fig. 2 The function-value-based spline fit in a 5-node window, $\epsilon = 0.5$



- (iii) The function-value base spline fit cannot preserve the convex shape or linear shape in the 5-node-window, no matter how long the horizontal length difference of convex corner is. It is the worst in shape preservation among the three types of spline fits for the convex corner.

Figure 2 shows the function-value-based spline fit for a convex corner with $\epsilon = 0.5$ in a 5-node window. We see that the spline fit vibrates in the first, second and fourth intervals, although very close to the line segments of corner. Obviously, neither the linear shape nor convex shape is preserved in this case. The reason may be that the function-value based cubic L^1 spline aims at minimizing the L^1 -approximation error in the whole window but then is unable to preserve the shape of convex corner.

Comparing the performances of three types of spline fits in the 5-node window suggests that the second-derivative based and first-derivative based spline fits are better in preserving the shape of convex corner. Moreover, the essential difference among these three types of spline fits may help understand their performances of shape-preservation for the convex corner. Note that a convex corner consists of two connective line segments with the slope increasing. The second-derivative-based spline fit finds a piecewise differentiable cubic spline with near-linear shape in each interval, hence has a strong capability of preserving the shape of convex corner. Compared with the second-derivative based spline fit, the first-derivative based spline fit further requires the near-linear shape close to the original linear shape in the convex corner. Therefore, the first-derivative based spline fit also has a strong capability of preserving the shape (especially the linear shape) of convex corner. When it comes to the function-value based spline fit, the unique criteria becomes the minimization of approximation error, i.e., finding a piecewise cubic spline whose function value is as close to that of convex corner as possible. Consequently, the function-value based spline fit neglects the other aspect of preserving the shape of convex corner.

6 Concluding Remarks

We have quantitatively characterized the shape-preserving performance of the second-derivative-based cubic L^1 spline fit for a convex corner. To calculate the spline fit, we

have developed an analytic approach in a 3-node window and a numerical procedure in a 5-node window. In both cases, the spline fit is shown to preserve the convex shape in the window as long as the horizontal length difference of two line segments is within the middle third of window's length. This study takes one step further from [19] and is of the first theoretic study to completely characterize the shape-preserving capability of second-derivative based spline fit for a convex shape.

Meanwhile, we have conducted numerical algorithms to compare the shape-preserving performance of three types of spline fits in a 5-node window. The second-derivative based and first-derivative based spline fits are shown better in preserving the shape of convex corner. This result facilitates understanding these three different types of spline fits for shape-preservation purpose. Comparing with [19], we observe that it may be difficult to approximate both a corner and a jump very well by using a single kind of L^1 spline fits. To hopefully overcome this issue, one practical recommendation is to use a mixed strategy which chooses the appropriate spline fit according to the data. When the data indicates discontinuity, for example a jump, the functional-value-based spline fit may be a good fit. When the data appears more like a corner, we may use second-derivative-based or first-derivative-based spline fit.

To fully evaluate the convex-shape-preserving capability, analytical solutions to the first-derivative based and function-value based in the 3-node window need to be derived. Furthermore, since the convex corner is only a simple one among all convex shapes, it is also worth investigating more complex convex shapes in the future.

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